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UNCERTAINTY BOUNDS FOR ENERGY OF IMPERFECTLY DESCRIBED VIBRATION FIELDS. APPLICATION TO THE VALIDITY OF TRUNCATED MODAL EXPANSION

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Abstract—Bounds of vibration energy are given in relation to a residual functional measuring the degree of approximation of the vibration fields, and a factor that can amplify or attenuate the uncertainty bounds. The amplification of energy uncertainty is considerable near resonance frequencies.

Assuming a statistical distribution of the error of description of the vibrations over the eigenmodes of the medium leads to a reduction in the uncertainty bounds, especially when the error is distributed over a large number of modes.

The bounds of energy can be used as a criterion of convergence for modal expansion. The application to acoustic response of a square room shows that, for a given modal expansion, the higher the energy level the better the convergence.

1. INTRODUCTION

The prediction of continuous media vibrations, consists of the calculation of stress and displacement fields resulting from forces excitations. Generally, approximations of these fields are obtained, after which the vibration state is imperfectly described. In such situations the energy calculated from approximate solutions, represents the exact vibration energy with uncertainties. The aim of this paper is to provide relations between uncertainty bounds of energy and a measure of the imperfection of vibration fields description.

The association of uncertainty bounds with approximate solution has especially been used to give bounds for eigenfrequencies (see, for example, Fichera, 1965; Weinstein and Stenger, 1972; Finlayson, 1972). In particular, the method used in this paper has been previously applied to eigenfrequencies' bounding by Guyader (1987). In contrast, application of this idea to vibration responses has not been used so much. However, it is possible to mention papers by Skudrzik (1980, 1987), on the bounding of input admittance on resonance and anti-resonance, and the works of Popplewell and Youssef (1979) and Popplewell *et al.* (1981), giving response maxima of vibrating systems excited by imperfectly known forces. The method used in this study is based on the following ideas: a continuous medium is considered, for which the exact solution perfectly describes the vibration behavior; this will be called "the reference problem". A functional is associated with each stress and displacement fields in order to measure how these fields verify the reference problem equations. [Works of Nayroles (1971) and Ladeveze (1975) have been used as a starting point to construct the functional.] Then the bounds of energy are related to the functional value.

The general case is treated first, then two specific situations are studied:

—The case where the imperfection of description can be distributed over the vibration modes following a probability density function.

—The case of normal mode expansion of solutions. A criterion of convergence for truncated expansion is given.

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The method is finally applied to the calculation of responses of rectangular rooms using normal mode expansion. The influence of the number of terms in the expansion is analyzed versus the excitation frequency.

2. THE REFERENCE PROBLEM

The problem of a three-dimensional viscoelastic continuous medium is considered. The governing equations for harmonic motion were derived by Mandel (1966).

Equation of motion

$$\Omega^2 \rho u_i + \sigma_{ij,j} + f_i = 0 \quad \text{in } V. \tag{1}$$

Stress–strain relation

$$\sigma_{ij} - (1 + j\eta) C_{ijkl} \varepsilon_{kl} = 0 \quad \text{in } V. \tag{2}$$

Boundary conditions

$$\sigma_{ij} n_j = 0 \quad \text{on } S_\sigma \tag{3}$$

$$u_i = 0 \quad \text{on } S_u. \tag{4}$$

Notation

V	volume of the continuous medium
Ω	driving angular frequency
u_i	displacement field
σ_{ij}	stress field
ε_{kl}	$= \frac{1}{2}(u_{k,l} + u_{l,k})$: strain field
n_j	components of the boundary normal unit vector
S	boundary surface $= S_u \cup S_\sigma$
S_σ	part of the boundary surface where stresses are zero
S_u	part of the boundary surface where displacements are zero
ρ	mass per unit volume
C_{ijkl}	elastic constants
f_i	driving force amplitude
η	material damping loss factor
j	$\sqrt{-1}$

(u_i, σ_{ij}, ε_{kl} and f_i, are complex quantities as a viscoelastic medium is considered).

Notes:

(a) The stress strain relation (2) corresponds to the case of Voigt isotopic material having equal damping loss factor for Young and Coulomb Moduli.
(b) We assume that the solution (σ_{ij}, u_i) is zero if and only if the driving force is zero. This hypothesis excludes the cases of pure elastic media at resonances and boundary conditions allowing rigid motion.

Let us introduce the eigenmodes of vibration, that we shall use later. We consider the elastic media obtained neglecting the damping in the stress–strain relation. The pth eigenmode is defined as the triplet (ω_p, u_i^p, σ_{ij}^p), which satisfies relations (5)–(8).

Equation of motion

$$\omega_p^2 \rho u_i^p + \sigma_{ij,j}^p = 0, \quad \text{in } V. \tag{5}$$

Stress–strain relation

$$\sigma^p_{ij} - C_{ijkl}\varepsilon^p_{kl} = 0, \quad \text{in } V. \tag{6}$$

Boundary conditions

$$\sigma^p_{ij} n_j = 0 \quad \text{on } S_\sigma, \tag{7}$$

$$u^p_i = 0 \quad \text{on } S_u. \tag{8}$$

All quantities in eqns (5)–(8) are now real.

Orthogonality properties of eigendisplacements and eigenstresses are given by the three basic relations (9)–(11):

$$\omega_p^2 \int_V \rho u^r_i u^p_i \, \mathrm{d}v = E_p \delta_{rp} \tag{9}$$

$$\int_V \sigma^r_{ij} C^{-1}_{ijkl} \sigma^p_{kl} \, \mathrm{d}v = E_p \delta_{rp} \tag{10}$$

$$\int_V \sigma^r_{ij} \varepsilon^p_{ij} \, \mathrm{d}v = E_p \delta_{rp} \tag{11}$$

with E_p being the norm of mode p, and

$$\delta_{rp} \begin{cases} = 1 & \text{if} \quad p = r \\ = 0 & \text{if} \quad p \neq r. \end{cases} \tag{12}$$

3. FORMULATION OF THE PROBLEM USING ENERGY RESIDUAL FUNCTIONAL

Let us introduce the two spaces: Σ, space of admissible complex stress fields and U, space of admissible complex displacement fields:

$$\Sigma = \{\tau_{ij} \in L_2(V) | \tau_{ij} n_j = 0 \quad \text{on} \quad S_\sigma\} \tag{13}$$

$$U = \{V_i \in H^1(V) | V_i = 0 \quad \text{on} \quad S_u\}. \tag{14}$$

The pair (u_i, σ_{ij}) solution of the problem (1)–(4) is element of the space product $U \times \Sigma$.

In general, a pair (V_i, τ_{ij}) does not verify eqns (1) and (2) and thus is an approximate solution of the problem. To estimate the quality of the approximation, we introduce a functional to measure the degree of verification of eqns (1) and (2) by the particular pair (V_i, τ_{ij}).

The stress–strain relation will be considered first. Following works of Nayroles (1971), Ladevcze (1975) and Guyader (1987), we define the energy residual functional ϕ_c (V_i, τ_{ij}) by:

$$\phi_c: \quad U \times \Sigma \rightarrow \mathbb{R}^+$$
$$(V_i, \tau_{ij}) \rightarrow \phi_c(V_i, \tau_{ij}) \tag{15}$$

$$\Phi_c(V_i, \tau_{ij}) = \int_V \mathrm{Re}\left\{(\tau_{ij} - (1 + j\eta) C_{ijkl} e_{kl}) \left(\frac{1 - j\eta}{1 + \eta^2}\right) C^{-1}_{ijpq} (\tau_{pq} - (1 + j\eta) C_{pqrs} e_{rs})^* \right\} \mathrm{d}v \tag{16}$$

with

$$\text{Re}\{\ \}:\ \text{real part}$$

$$(\ )^*:\ \text{complex conjugate}$$

$$e_{kl} = \tfrac{1}{2}(V_{k,l} + V_{l,k}). \tag{17}$$

The functional ϕ_c is a quadratic form; and verifies the relations:

$$\phi_c(V_i, \tau_{ij}) = 0 \Leftrightarrow \tau_{ij} = C_{ijkl}e_{kl} \tag{18}$$

when

$$\phi_c(V_i, \tau_{ij}) > 0 \Leftrightarrow \tau_{ij} \neq C_{ijkl}e_{kl}. \tag{19}$$

From a physical point of view, the functional represents a residual energy resulting from the imperfect satisfaction of the stress–strain relation (2).

In the same way, the functional $\phi_M(V_i, \tau_{ij})$ is introduced in order to measure the error of satisfaction of the equation of motion (1):

$$\phi_M(V_i, \tau_i) = \int_V \left\{(\tau_{ij,j} + \rho\Omega^2 V_i + f_i)\frac{1}{\rho\Omega^2}(\tau_{ij,j} + \rho\Omega^2 V_i + f_i)^*\right\} \mathrm{d}v. \tag{20}$$

This functional is a residual kinetic energy, giving the degree of fulfilment, by a pair (V_i, τ_{ij}) of the equation of motion (1):

$$\phi_M(V_i, \tau_{ij}) = 0 \Leftrightarrow \tau_{ij,j} + \rho\Omega^2 V_i + f_i = 0 \quad \text{in } V \tag{21}$$

$$\phi_M(V_i, \tau_i) > 0 \Leftrightarrow \tau_{ij,j} + \rho\Omega^2 V_i + f_i \neq 0 \quad \text{in } V. \tag{22}$$

The global functional:

$$\phi(V_i, \tau_{ij}) = \phi_c(V_i, \tau_{ij}) + \phi_M(V_i, \tau_{ij}) \tag{23}$$

is a quadratic form, greater or equal to zero, such that:

$$\phi(V_i, \tau_{ij}) = 0 \Leftrightarrow \begin{cases} \tau_{ij,j} + \Omega^2 \rho V_i + f_i = 0 & \text{in } V \\ \tau_{ij} - C_{ijkl}e_{kl} = 0 & \text{in } V. \end{cases} \tag{24}$$

To find the solution of the reference problem consists in the determination of the stress and displacement fields satisfying the boundary conditions and annulling the functional $\phi(V_i, \tau_{ij})$. However, from a mathematical point of view, it is simpler to minimize a functional than to annul it.

Thus Guyader (1987) introduced a formulation for the reference problem using the residual functional ϕ:

$$\text{find the pair } (u_i, \sigma_{ij}) \in U \times \Sigma, \quad \text{such that:}$$

$$\phi(u_i, \sigma_{ij}) = \min \phi(V_i, \tau_i). \tag{25}$$

From a physical point of view, the solution is now obtained by looking for a smallest residual as possible; if zero is reached the exact solution is found, while if the value of the functional remains greater than zero an approximate solution is obtained.

The quality of an approximation can be estimated from the functional value, the smaller the residual the better the approximation. Moreover it is possible to relate energy uncertainty bounds to the functional value, as will be shown in the next section.

4. KINETIC ENERGY UNCERTAINTY BOUNDS OF IMPERFECTLY DESCRIBED DISPLACEMENT FIELDS

Let us introduce the following new functional, which, in fact, is a norm in the space product $U \times \Sigma$:

$$\| V_i, \tau_{ij} \|^2 = \int_V \mathrm{Re}\left\{(\tau_{ij} - C_{ijkl}(1+j\eta)e_{kl})\frac{1-j\eta}{1+\eta^2} C^{-1}_{ijpq}(\tau_{pq} - C_{pqrs}(1+j\eta)e_{rs})^*\right\} \mathrm{d}v$$
$$+ \int_V \left\{(\tau_{ij,j} + \rho\Omega^2 V_i)\frac{1}{\rho\Omega^2}(\tau_{ij,j} + \rho\Omega^2 V_i)^*\right\} \mathrm{d}v. \quad (26)$$

There is an obvious relationship between the norm (26) and the residual functional $\phi(V_i, \tau_{ij})$; it will be used in the following.

The stress and displacement fields are expanded over the eigenstress and eigendisplacement fields of the elastic medium:

$$\tau_{ij} = \sum_{n=1}^{\infty} a_n \tau^n_{ij} \quad (27)$$

$$V_i = \sum_{n=1}^{\infty} b_n u^n_i. \quad (28)$$

a_n and b_n are complex numbers. After substitution of modal expansion (27) and (28) into (26) and using (9), (10) and (11), we obtain:

$$\| V_i, \tau_{ij} \|^2 = \sum_{n=1}^{\infty} \left(\left| a_n - (1+j\eta)b_n \right|^2 \Big/ (1+\eta^2) + \left| a_n \frac{\omega_n}{\Omega} - b_n \frac{\Omega}{\omega_n} \right|^2 \right) E_n. \quad (29)$$

E_n: is the norm of the nth mode.

Let us look now at the kinetic energy of the displacement field V_i:

$$T(V_i) = \int_V \rho\Omega^2 |V_i|^2 \, \mathrm{d}v. \quad (30)$$

Introduction of the modal expansion (28) into (30) and use of orthogonality properties gives, after calculations:

$$T(V_i) = \sum_{n=1}^{\infty} |b_n|^2 \frac{\Omega^2}{\omega_n^2} E_n. \quad (31)$$

In the Appendix, the following lower bound of the norm is obtained:

$$\| V_i, \tau_{ij} \|^2 \geqslant \sum_{n=1}^{\infty} |b_n|^2 \frac{\Omega^2}{\omega_n^2} \Gamma_n E_n \quad (32)$$

with

$$\Gamma_n = \frac{\omega_n^2}{\Omega^2}\left(\left| \frac{\Omega}{\omega_n} - \frac{\omega_n}{\Omega}\alpha_n \right|^2 + \left| \alpha_n - 1 - j\eta \right|^2 \right) \quad (33)$$

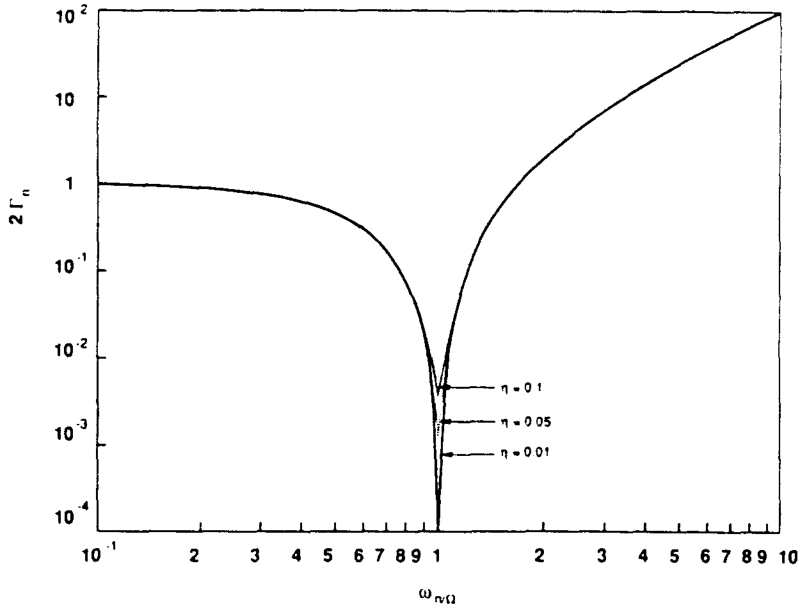

Fig. 1. Variation of function Γ_n versus Ω/ω_n for different damping loss factors.

and

$$\alpha_n = \frac{2 + j\eta + \eta^2}{\left[\frac{\omega_n^2}{\Omega^2}(1+\eta^2) + 1\right]}. \tag{34}$$

Comparison of expressions (32) and (31) shows that the lower bound of the norm is the sum of the modal kinetic energies weighted by the factor Γ_n. When $\Gamma_n > 1$, the corresponding mode has an influence on the norm lower bound, amplified compared to that on the kinetic energy. When $\Gamma_n < 1$ the opposite situation is observed. When $\Gamma_n = 1$ the mode has the same influence on the kinetic energy and the norm lower bound.

The weighting function Γ_n is plotted versus Ω/ω_n in Fig. 1. The three previous situations are possible depending on the driving frequency if:

$$\omega_n < \Omega \quad \Gamma_n \to 1 \tag{35}$$

if

$$\omega_n \approx \Omega \quad \Gamma_n \to \eta^2/2 \Rightarrow \Gamma_n < 1 \tag{36}$$

if

$$\omega_n > \Omega \quad \Gamma_n \to \left(\frac{\omega_n}{\Omega}\right)^2 \Rightarrow \Gamma_n > 1. \tag{37}$$

In order to relate the kinetic energy and the norm, let us define the factor Γ as:

$$\Gamma = \operatorname*{Min}_{n=1,\infty} \Gamma_n. \tag{38}$$

Using (31), (32) and (38) gives an upper bound for kinetic energy

$$T(V_i) \leqslant \| V_i, \tau_{ij} \|^2 / \Gamma. \tag{39}$$

This inequality is the basis for the bounding of the exact kinetic energy.

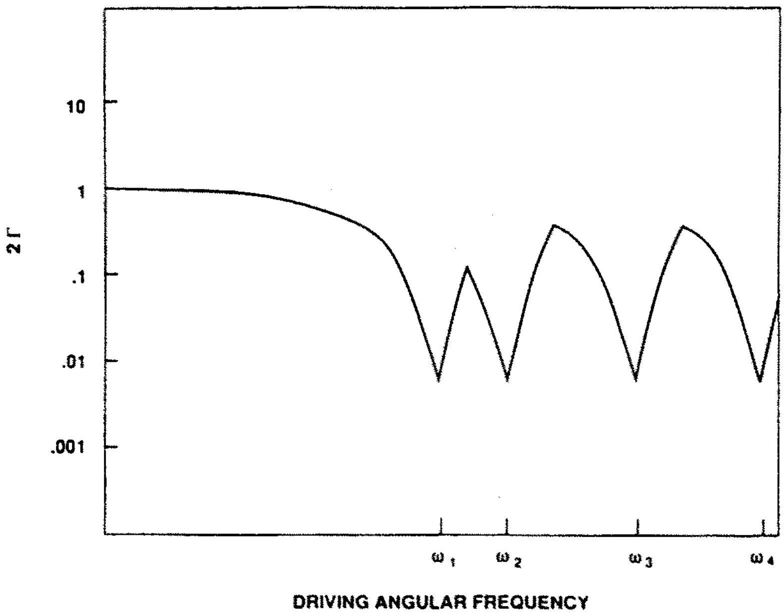

Fig. 2. Variation of factor Γ with frequency.

The stress and displacement fields considered previously are written as the difference between the exact (u_i, σ_{ij}) and the approximate solutions: $(\hat{u}_i, \hat{\sigma}_{ij})$

$$\tau_{ij} = \sigma_{ij} - \hat{\sigma}_{ij} \tag{40}$$

$$V_i = u_i - \hat{u}_i. \tag{41}$$

It follows from (39), (40) and (41), that the following relation holds:

$$T(u_i - \hat{u}_i) \leqslant \frac{\|\sigma_{ij} - \hat{\sigma}_{ij}, u_i - \hat{u}_i\|^2}{\Gamma}. \tag{42}$$

A simple calculation shows that

$$\|\sigma_{ij} - \hat{\sigma}_{ij}, u_i - \hat{u}_i\|^2 = \phi(\hat{u}_i, \hat{\sigma}_{ij}). \tag{43}$$

[One needs only to use (40) and (41) in (26) and remember that the exact solution verifies (1) and (2).] It follows from (42) and (43) that:

$$T(u_i - \hat{u}_i) \leqslant \phi(\hat{u}_i, \hat{\sigma}_{ij})/\Gamma. \tag{44}$$

The kinetic energy defined in (30) is the square of the norm of $L_2(U)$. Thus one has:

$$\sqrt{T(\hat{u}_i)} - \sqrt{T(\hat{u}_i - u_i)} \leqslant \sqrt{T(u_i)} \leqslant \sqrt{T(\hat{u}_i)} + \sqrt{T(\hat{u}_i - u_i)}. \tag{45}$$

Then replacing $T(\hat{u}_i - u_i)$ with (44), one obtains:

$$\sqrt{T(\hat{u}_i)} - \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{\Gamma}} \leqslant \sqrt{T(u_i)} \leqslant \sqrt{T(\hat{u}_i)} + \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{\Gamma}}. \tag{46}$$

These bounds on the kinetic energy depend only on the approximate solution $(\hat{\sigma}_{ij}, \hat{u}_i)$ and on the factor Γ: relation (46) gives an energy uncertainty range related to the imperfection of description of the vibration fields as measured by the residual functional.

Inequalities (46) show that the smaller the residual functional the narrower the bounds; however, the factor Γ also plays an important role. Typical variations of the factor Γ versus frequency are plotted in Fig. 2. One can see that Γ is strongly variable with frequency and takes a minimum value equal to $\eta^2/2$, at resonance frequencies. As a consequence, for a given measure of the imperfection of the vibration fields description (that is to say for a given functional value), the energy uncertainty range is strongly amplified near resonance frequencies, especially for low structural damping.

From a physical point of view, this tendency may be understood as a high sensitivity of the vibration response of structures at resonances.

In the most general case, since the eigenfrequencies of the continuous medium are unknown, it is then necessary to take into account the smaller value of Γ whatever the frequency, that is to say $\eta^2/2$. The uncertainty range is then amplified at each frequency especially for weak damping. The bounds of kinetic energy now depend only on the approximate solution $(\hat{\sigma}_{ij}, \hat{u}_i)$ and the structural loss factor η:

$$\sqrt{T(\hat{u}_i)} - \frac{\sqrt{2\phi(\hat{u}_i, \hat{\sigma}_{ij})}}{\eta} \leqslant \sqrt{T(u_i)} \leqslant \sqrt{T(\hat{u}_i)} + \frac{\sqrt{2\phi(\hat{u}_i, \hat{\sigma}_{ij})}}{\eta}. \tag{47}$$

5. UNCERTAINTY BOUNDS OF DEFORMATION ENERGY FOR IMPERFECTLY DESCRIBED STRESS FIELDS

The calculations are presented briefly, since the same procedure as in Section 4 for kinetic energy must be employed in the present case. The deformation energy $E(\tau_{ij})$ of the stress field $\tau_{ij} \in \Sigma$ is given by:

$$E(\tau_{ij}) = \int_V \mathrm{Re}\,\{\tau_{ij} C_{ijkl}^{-1*} \tau_{kl}^*\}\, dv. \tag{48}$$

Insertion of the modal expansion (27) in this expression, gives after calculations:

$$E(\tau_{ij}) = \sum_{n=1}^{\infty} |a_n|^2 \frac{E_n}{1+\eta^2}. \tag{49}$$

In the Appendix, the following result is demonstrated:

$$\| V_i, \tau_{ij} \|^2 \geqslant \sum_{n=1}^{\infty} \Delta_n |a_n|^2 \frac{E_n}{1+\eta^2} \tag{50}$$

with

$$\Delta_n = |1-(1+j\eta)\beta_n|^2/(1+\eta^2) + \left|\frac{\omega_n}{\Omega} - \beta_n \frac{\Omega}{\omega_n}\right|^2 \tag{51}$$

and

$$\beta_n = \frac{2+\eta^2+j\eta}{1+2j\eta-\eta^2+(1+\eta^2)\dfrac{\Omega^2}{\omega_n^2}}. \tag{52}$$

Comparison of eqns (49) and (50) shows that inequality (53) holds:

$$E(\tau_{ij}) \leqslant \| V_i, \tau_{ij} \|^2/\Delta \tag{53}$$

with

$$\Delta = \operatorname*{Min}_{n=1,\infty} \Delta_n. \tag{54}$$

The previous relation (53) is formally identical to eqn (39) which is the basis for bounding kinetic energy. Thus, following similar calculations as in Section 4, one obtains:

$$\sqrt{E(\hat{\sigma}_{ij})} - \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{\Delta}} \leqslant E(\sigma_{ij}) \leqslant \sqrt{E(\hat{\sigma}_{ij})} + \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{\Delta}}. \tag{55}$$

Bounds of deformation energy have the same form as bounds for kinetic energy (46), Γ is just replaced by Δ. In addition, a comparison of Δ_n and Γ_n shows that both quantities take close values, in particular their minima are equal to $\eta^2/2$.

In the general case, since the eigenfrequencies of the continuous media are unknown, the following bounds of deformation energy are obtained:

$$\sqrt{E(\hat{\sigma}_{ij})} - \frac{\sqrt{2\phi(\hat{u}_i, \hat{\sigma}_{ij})}}{\eta} \leqslant E(\sigma_{ij}) \leqslant \sqrt{E(\hat{\sigma}_{ij})} + \frac{\sqrt{2\phi(\hat{u}_i, \hat{\sigma}_{ij})}}{\eta}. \tag{56}$$

The similarity of the results for kinetic and deformation energy makes it unnecessary to study both cases, and in the following only kinetic energy will be studied, the extension to deformation energy being obvious.

6. PROBABILISTIC ASPECT FOR UNCERTAINTY BOUNDS OF KINETIC ENERGY

From a physical point of view the kinetic energy bounds (47) are reached when the error of description is concentrated on one mode excited at resonance, that is to say:

$$\begin{cases} \sigma_{ij} - \hat{\sigma}_{ij} = b_p \sigma_{ij}^p \\ u_i - \hat{u}_i = a_p u_i^p \end{cases} \quad \text{and} \quad \Omega = \omega_p. \tag{57}$$

From a statistical point of view, such a situation is very improbable, and generally the error of description is distributed over all modes. Then

$$T(u_i - \hat{u}_i) = \sum_{n=1}^{\infty} |b_n|^2 \frac{\Omega^2}{\omega_n^2} E_n = \sum_{n=1}^{\infty} T_n(u_i - \hat{u}_i) \tag{58}$$

where $T_n(u_i - \hat{u}_i)$ is the modal kinetic energy of the difference between exact and approximate displacement fields.

Let us define the ratio h_n, of modal to total error of kinetic energy:

$$h_n = \frac{T_n(u_i - \hat{u}_i)}{T(u_i - \hat{u}_i)}. \tag{59}$$

It is obvious that h_n is a real number, verifying:

$$0 < h_n < 1 \tag{60}$$

and

$$\sum_{n=1}^{\infty} h_n = 1. \tag{61}$$

Using (59) and (32), one obtains:

$$\|\sigma_{ij}-\hat{\sigma}_{ij}, u_i-\hat{u}_i\|^2 \geqslant \sum_{n=1}^{\infty} \Gamma_n h_n T(u_i-\hat{u}_i). \tag{62}$$

Then, replacing the norm by the residual functional as stated in (43), gives:

$$T(u_i-\hat{u}_i) \leqslant \phi(\hat{u}_i, \hat{\sigma}_{ij})/\gamma \tag{63}$$

with

$$\gamma = \sum_{n=1}^{\infty} \Gamma_n h_n. \tag{64}$$

Expression (63) is similar to (44), and as a consequence we obtain the kinetic energy bounds in the same way as in Section 4:

$$\sqrt{T(\hat{u}_i)} - \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{\gamma}} \leqslant \sqrt{T(u_i)} \leqslant \sqrt{T(\hat{u}_i)} + \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{\gamma}}. \tag{65}$$

The value of γ depends on that of h_n, that is to say on the distribution over the modes of the error on kinetic energy. This distribution depends on the case of interest, however it is possible to give an expected value of γ, replacing the previous deterministic presentation by the following statistical approach.

Let us consider the set Ω of the modes of the continuous medium and define the probability h'_n of mode n to participate in the error on kinetic energy with $0 < h'_n < 1$:

$$\sum_{n=1}^{\infty} h'_n = 1.$$

Let us introduce the random variable γ':

$$\gamma' : \Omega \longrightarrow \mathbb{R}$$

$$\text{mode } n \longrightarrow \gamma'(\text{mode } n) = \Gamma_n.$$

The expected value of γ' is

$$E(\gamma') = \sum_{n=1}^{\infty} \Gamma_n h'_n. \tag{66}$$

The expressions γ and $E(\gamma')$ are similar, the ratio h_n being replaced by the probability h'_n. Thus, $E(\gamma')$ can be interpreted as a statistical estimation of γ, and the following statistical estimation for bounds of kinetic energy are obtained:

$$\sqrt{T(\hat{u}_i)} - \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{E(\gamma')}} \leqslant \sqrt{T(u_i)} \leqslant \sqrt{T(\hat{u}_i)} + \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{E(\gamma')}}. \tag{67}$$

As an example, we assume that the probability h'_n is distributed following Poisson's law and that the continuous media is a simply supported beam in flexural motion. Then the eigen angular frequencies are given by:

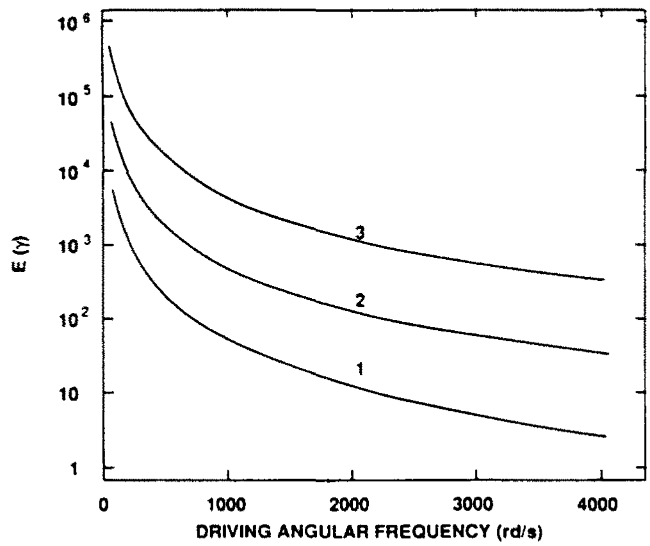

Fig. 3. Variation of $E(\gamma')$ versus driving frequency for three values of the parameter λ of Poisson's law. Curve 1: $\lambda = 5$; curve 2: $\lambda = 10$; curve 3: $\lambda = 19$.

$$\omega_n = \sqrt{\frac{Eh^2}{12\rho}} \cdot \frac{n^2\Pi^2}{L^2}$$

where E is the Young's Modulus and ρ the mass by unit volume of the beam material, L is the length and h the thickness of the beam.

Figure 3 shows the expected value $E(\gamma')$ versus the driving angular frequency. Three curves are presented for different values of the parameter λ of Poisson's law. The mean value and the standard deviation of Poisson's law are respectively equal to λ and $\sqrt{\lambda}$, from a physical point of view a small value of λ indicates that the error is concentrated over a few modes of low resonance frequencies, a large value indicates that the error is distributed over a large number of modes situated at higher frequency. It can be noticed, from Fig. 3 that the following tendencies apply:

—at a given frequency, $E(\gamma')$ decreases with λ indicating that the bounds on energy increase when the error is concentrated over a smaller number of modes;

—$E(\gamma')$ is a decreasing function of the frequency, and tends to unity. $E(\gamma')$ is then considerably greater than $\eta^2/2$ and the statistical distribution of the error of description provides a strong reduction of the uncertainty range of kinetic energy when compared to the deterministic bounds (47).

7. UNCERTAINTY BOUNDS OF KINETIC ENERGY FOR APPROXIMATE SOLUTIONS OBTAINED BY TRUNCATED MODAL EXPANSIONS

The expansion of solutions over the normal modes of vibration is a widely used method to approximate the displacements and stress fields. The main problem that arises when applying the method is the choice of the modes that have to be taken into account to give good approximations. A criterion of convergence is obtained using the bounds of kinetic energies to estimate the quality of the approximate solution.

Let us consider an approximate solution $(\hat{\sigma}_{ij}, \hat{u}_i)$ obtained by truncated modal expansion:

$$\hat{\sigma}_{ij} = \sum_{n \in I} a_n \sigma^n_{ij} \tag{68}$$

$$\hat{u}_i = \sum_{n \in I} b_n u^n_i . \tag{69}$$

I is the set of the indices of modes taken into account in the calculation of the approximate solutions. In general, the following two particular sets are considered:

(a) I' is the set of indexes of modes for which resonance frequencies are located in a band around the excitation angular frequency:

$$I' = \{n \in \mathbb{N} | \omega_n \in [\Omega_0, \Omega_1]\}, \tag{70}$$

(b) I'_0 is a particular case of set I', obtained by setting $\Omega_0 = 0$ in order to take into account all the modes for which resonance angular frequency are lower than Ω_1:

$$I'_0 = \{n \in \mathbb{N} | \omega_n \in [0, \Omega_1]\}. \tag{71}$$

To determine the kinetic energy uncertainty bounds associated with approximate solutions (68) and (69), it is possible to use the inequalities (46). However these bounds are generally large and for particular approximate solutions, narrower bounds can be obtained.

Let us consider the differences between the exact and the truncated modal expansion of stress and displacement fields:

$$\sigma_{ij} - \hat{\sigma}_{ij} = \sum_{n \in \mathbb{N} - I} a_n \sigma_{ij}^n \tag{72}$$

$$u_i - \hat{u}_i = \sum_{n \in \mathbb{N} - I} b_n u_i^n. \tag{73}$$

Introducing (72) and (73) into inequality (32) gives

$$\|\sigma_{ij} - \hat{\sigma}_{ij}, u_i - \hat{u}_i\|^2 \leqslant \sum_{n \in \mathbb{N} - I} \Gamma_n |b_n|^2 \frac{\Omega^2}{\omega_n^2} E_n. \tag{74}$$

It follows, using (43), that

$$\phi(\hat{u}_i, \hat{\sigma}_{ij}) \leqslant \delta \sum_{n \in \mathbb{N} - I} |b_n|^2 \frac{\Omega^2}{\omega_n^2} E_n = \delta T(u_i - \hat{u}_i) \tag{75}$$

with

$$\delta = \underset{n \in \mathbb{N} - I}{\operatorname{Min}} \Gamma_n. \tag{76}$$

This inequality (75) is equivalent to (44); thus we obtain, in the same way as in Section 4, the bounds:

$$\sqrt{T(\hat{u}_i)} - \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{\delta}} \leqslant \sqrt{T(u_i)} \leqslant \sqrt{T(\hat{u}_i)} + \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{\delta}}. \tag{77}$$

(77) is formally equivalent to (46) but δ is always greater than Γ, thus, for a given measure of the error of description by the functional $\phi(\hat{u}_i, \hat{\sigma}_{ij})$, the uncertainty bounds are narrower for modal expansion approximation than for the general case.

Let us now examine the values of δ for the particular sets I' and I'_0:

$$\delta' = \operatorname*{Min}_{n \in \mathbb{N} - I} \Gamma_n \tag{78}$$

$$\delta'_0 = \operatorname*{Min}_{n \in \mathbb{N} - I_0} \Gamma_n. \tag{79}$$

After calculation one obtains:

$$\delta' = \min_{i=0,1} \left[\left(\frac{\Omega_i}{\Omega}\right)^2 \left(\left|\frac{\Omega}{\Omega_i} - \frac{\Omega_i}{\Omega}\alpha_i\right|^2 + \frac{1}{1+\eta^2}\left|\alpha_i - 1 - j\eta\right|^2 \right) \right] \tag{80}$$

with

$$\alpha_i = (2 + j\eta + \eta^2) \Big/ \left(\left(\frac{\Omega_i}{\Omega}\right)^2 (1+\eta^2) + 1 \right) \tag{81}$$

$$\delta'_0 = \left[\left(\frac{\Omega_1}{\Omega}\right)^2 \left(\left|\frac{\Omega}{\Omega_1} - \frac{\Omega_1}{\Omega}\alpha_1\right|^2 + \frac{1}{1+\eta^2}\left|\alpha_1 - 1 - j\eta\right|^2 \right) \right]. \tag{82}$$

An examination of expressions (80)–(82) shows the two basic tendencies:

(a) δ' increases as Ω_0 decreases and/or Ω_1 increases, that is when the number of modes taken into account for the calculation increases. The same tendency remains true for δ'_0 that increases with Ω_1. So, when the number of modes taken into account increases, the kinetic energy uncertainty bounds become narrower as the result of both decrease of the residual functional and increase of δ' (resp. δ'_0).

(b) The asymptotic values obtained in the case $\Omega_1 > \Omega$ are:

$$\delta' = 1 \tag{83}$$

$$\delta'_0 = \left(\frac{\Omega_1}{\Omega}\right)^2 > 1. \tag{84}$$

δ'_0 is thus much greater than δ', and for a given value of the residual functional, the approximation using the set I_0 gives smaller uncertainty bounds than the approximation using I.

8. MODAL EXPANSION OF THE ACOUSTIC RESPONSE OF A ROOM

Let us consider a square room with Neumann boundary conditions. The acoustic pressure p and displacement X_i, solutions of the problem must satisfy the following equations:

$$\rho\Omega^2 X_i + p_{,i} = 0 \quad \text{in } V \tag{85}$$

$$p + \rho c^2(1 + j\eta)X_{i,i} = s \quad \text{in } V \tag{86}$$

$$X_n = 0 \quad \text{on } S, \tag{87}$$

with

- V: volume of the room
- S: boundary surface of the room
- ρ: mass per unit volume of acoustic medium
- c: velocity of sound
- η: damping loss factor

s: acoustic sources
X_n: normal acoustic displacement on boundary.

This problem is formally identical to the reference problems (1)–(4), and therefore introducing the residual functional (88), the results of previous sections can be applied. In particular the bounds (77) can be used to study the validity of truncated modal expansion

$$\phi(p, X_i) = \int_V \left(|-\rho\Omega^2 X_i + p_{,i}|^2/\rho\Omega^2 + |p + \rho c^2(1 + j\eta)X_{i,i} - s|^2 \frac{1}{\rho c^2(1+\eta^2)} \right) \mathrm{d}v. \quad (88)$$

The solution of problems (87)–(89), when expanded over eigenmodes, takes the form:

$$\hat{p} = \sum\sum\sum_{(n,m,r)\in I} A_{nmr} \cos\frac{n\pi}{L_1}x_1 \cos\frac{m\pi}{L_2}x_2 \cos\frac{r\pi}{L_3}x_3 \quad (89)$$

$$\hat{X}_1 = \sum\sum\sum_{(n,m,r)\in I} \frac{A_{nmr}}{\rho\Omega^2}\frac{n\pi}{L_1} \sin\frac{n\pi}{L_1}x_1 \cos\frac{m\pi}{L_2}x_2 \cos\frac{r\pi}{L_3}x_3 \quad (90)$$

$$\hat{X}_2 = -\sum\sum\sum_{(n,m,r)\in I} \frac{A_{nmr}}{\rho\Omega^2}\frac{m\pi}{L_2} \cos\frac{n\pi}{L_1}x_1 \sin\frac{m\pi}{L_2}x_2 \cos\frac{r\pi}{L_3}x_3 \quad (91)$$

$$\hat{X}_3 = -\sum\sum\sum_{(n,m,r)\in I} \frac{A_{nmr}}{\rho\Omega^2}\frac{r\pi}{L_3} \cos\frac{n\pi}{L_1}x_1 \cos\frac{m\pi}{L_2}x_2 \sin\frac{r\pi}{L_3}x_3 \quad (92)$$

with

$$A_{nmr} = \frac{\varepsilon_n\varepsilon_m\varepsilon_r}{\left[1-(1+j\eta)\left(\frac{\omega_{nmr}}{\Omega}\right)^2\right]} \times \int_0^{L_1}\int_0^{L_2}\int_0^{L_3} \cos\frac{n\pi}{L_1}x_1 \cos\frac{m\pi}{L_2}x_2 \cos\frac{r\pi}{L_3}x_3 s(x_1, x_2, x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \quad (93)$$

$$\varepsilon_n \begin{cases} 1 & \text{if} \quad n = 0 \\ 2 & \text{if} \quad n \neq 0 \end{cases}$$

ω_{nmr}: eigen angular frequency of the room mode (n, m, r)
L_1, L_2, L_3: dimensions of the room
I: set of indexes (n, m, r).

Let us consider a pulsating rectangular source of dimensions (a_1, a_2, a_3) located at (z_1, z_2, z_3).

Introduction of the approximate solutions $(\hat{p}, \hat{X}_i)$ into the residual functional (88) gives, after calculations:

$$\phi(\hat{p}, \hat{X}_i) = \left(a_1a_2a_3 - \sum\sum\sum_{(n,m,r)\in I} \frac{\varepsilon_n\varepsilon_m\varepsilon_r}{L_1L_2L_3}(S_1(n)S_2(m)S_3(r)) \right)^2 \Big/ \rho c^2(1+\eta^2) \quad (94)$$

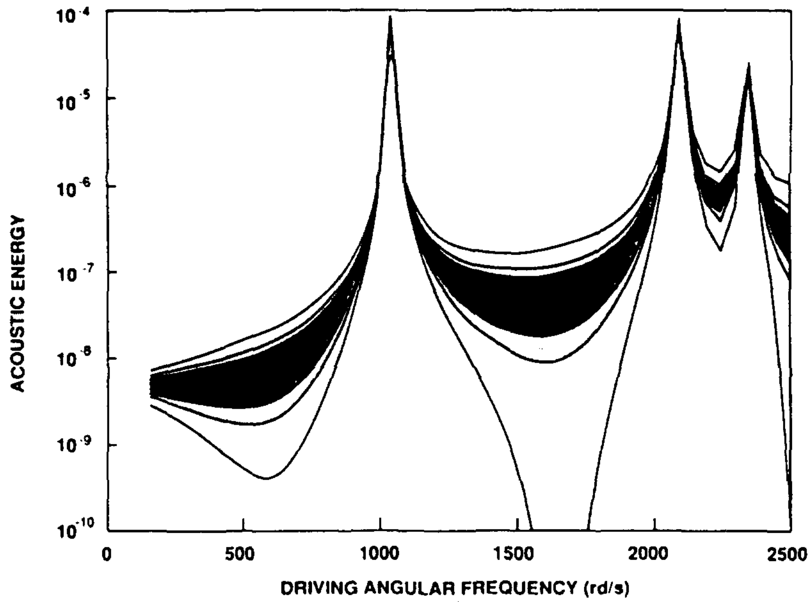

Fig. 4. Room acoustic energy versus excitation frequency for three modal expansions (Ω_1 = 4000, 6000, 8000 rd s^{-1}). Characteristics of the room: $L_1 = L_2 = L_3 = 1$ m; $\eta = 0.01$. Characteristics of the source: $(z_1, z_2, z_3) = (0.12, 0.5, 0.5)$; $(a_1, a_2, a_3) = (0.02, 0.04, 0.04)$.

with

$$\begin{cases} S_i(n) = \dfrac{L_i}{n\pi}\left(\sin\dfrac{n\pi}{L_i}\left(z_i + \dfrac{a_i}{2}\right) - \sin\dfrac{n\pi}{L_i}\left(z_i - \dfrac{a_i}{2}\right)\right) & \text{if } n \neq 0. \\ S_i(0) = a_i \end{cases} \tag{95}$$

It can be seen from expression (94), that the residual functional decreases and goes to zero as the set I increases.

The approximate acoustic energy of the room is given by:

$$E(\dot{p}) = \int_V \frac{|\dot{p}|^2}{\rho c^2(1+\eta^2)}\,dv = \frac{1}{\rho c^2(1+\eta^2)} \sum\sum\sum_{(n,m,r)\in I} |A_{nmr}|^2 \frac{L_1 L_2 L_3}{\varepsilon_n \varepsilon_m \varepsilon_r}. \tag{96}$$

If the set of indices I is chosen in order to be an I'_0 set, then:

$$I = I'_0 = \left\{(n,m,r) \in \mathbb{N}^3 \,\middle|\, \frac{c}{\pi}\sqrt{\frac{n^2}{L_1^2} + \frac{m^2}{L_2^2} + \frac{r^2}{L_3^2}} < \Omega_1\right\}. \tag{97}$$

The bounds (77) can be used to give uncertainty bounds of acoustic energy, associated with a truncated modal expansion.

Figure 4 shows the influence of the number of modes taken into account in modal expansions of the acoustic energy.

At resonances the acoustic energy is almost perfectly described, especially when the excitation angular frequency is well below Ω_1. Between two resonances the energy uncertainty is large, especially when the excitation angular frequency is close to Ω_1.

As a general rule it can be said that for a given modal expansion, the higher the energy level, the better the description.

9. REMARKS

(a) *Local energy*. Bounds have been given for the energy of the whole medium. If one is interested in local energy, however, for example local kinetic energy $t(V_i)$:

$$t(V_i) = \int_{v'} \rho\Omega^2 |V_i|^2 \, dv, \quad \text{with } v' \subset V$$

$$v': \quad \text{local volume,} \tag{98}$$

it is possible to obtain bounds in the following way.

First let us remark that the local energy is less than or equal to the total energy:

$$t(V_i) \leqslant T(V_i).$$

Then, applying this inequality to the particular displacement field,

$$V_i = u_i - \hat{u}_i$$

and using (44) one obtains:

$$t(u_i - \hat{u}_i) \leqslant \phi(\hat{u}_i, \hat{\sigma}_{ij})/\Gamma. \tag{99}$$

In the same way as in Section 4 this gives the bounds for local kinetic energy:

$$\sqrt{t(\hat{u}_i)} - \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{\Gamma}} \leqslant \sqrt{t(u_i)} \leqslant \sqrt{t(\hat{u}_i)} + \sqrt{\frac{\phi(\hat{u}_i, \hat{\sigma}_{ij})}{\Gamma}}. \tag{100}$$

These bounds show that the absolute uncertainty for local or total energies is the same, and that the relative uncertainty is greater for local energy. This is consistent with experimental results, as it is more difficult to measure local quantities precisely than it is to measure global ones.

In addition it can be said that it is impossible to find narrower bounds than (100) as, in the general case, the error of description can be concentrated in the local volume considered.

(b) *The case of non-proportional damping material.* To derive the energy bounds we assume a particular case of viscoelastic material satisying the stress–strain relation (2), i.e. an isotropic Voigt material with equal loss factors for Young and Coulomb moduli. For more complicated materials the problem is even greater to solve, as orthogonality properties of elastic eigenmodes are not true for such materials.

However it is possible to simply obtain bounds using the fact that for a given system, an increase of damping decreases the response and thus kinetic and elastic energy. For example, let us consider an isotopic Voigt material with different damping loss factors for Young and Coulomb moduli:

$$E(1 + j\eta_E) \tag{101}$$

$$G(1 + j\eta_G). \tag{102}$$

The stress–strain relation of that material is not of the form given in eqn (2), and the exact kinetic energy $T(u_i)$ cannot be bounded directly with (47).

Nevertheless, let us introduce the two associated problems, with same equation of motion (1) and boundary conditions (3) and (4) but different stress-strain relations:

$$\sigma_{ij} = (1 + j\eta_E) C_{ijkl} \varepsilon_{kl} \tag{103}$$

$$\sigma_{ij} = (1 + j\eta_G) C_{ijkl} \varepsilon_{kl}. \tag{104}$$

Noting $T_E(u_i^E)$ [resp. $T_G(u_i^G)$], the kinetic energy of the exact solution of the problem with stress–strain relation (103), [respectively (104)]; and assuming for example $\eta_E > \eta_G$, one can say:

$$T_E(u_i^E) \geqslant T(u_i) \geqslant T_G(u_i^G). \quad (105)$$

As the stress–strain relations (103) and (104) are of the form of eqn (2), it is possible to bound $T_E(u_i^E)$ and $T_G(u_i^G)$ with (47), and then use (105) to obtain:

$$\sqrt{T_E(\hat{u}_i)} - \frac{\sqrt{2\phi_E(\hat{u}_i, \hat{\sigma}_{ij})}}{\eta_E} \leqslant \sqrt{T(u_i)} \leqslant \sqrt{T_G(\hat{u}_i)} + \frac{\sqrt{2\phi_G(\hat{u}_i, \hat{\sigma}_{ij})}}{\eta_G} \quad (106)$$

with $(\hat{\sigma}_{ij}, \hat{u}_i)$, approximate solution and $\phi_E(\hat{u}_i, \hat{\sigma}_{ij})$ [resp. $\phi_G(\hat{u}_i, \hat{\sigma}_{ij})$], residual functional built with stress–strain relation [(103), (resp. (104)].

10. CONCLUSION

This work provides a method of bounding the kinetic and displacement energies of a viscoelastic solid using a residual functional associated with statistically and kinematically admissible stress and displacement fields. Compared to previous works of Skudrzik (1980, 1987) and Popplewell and Youssef (1979) and Popplewell *et al.* (1981), our method is general and gives the possibility of introducing statistical bounds and studying particular types of approximate solution like modal expansion.

In this paper, the general case was studied first showing that the bounds on energy depend on the value of residual functional and a factor strongly variable with frequency (at resonance frequencies the bounds are amplified, especially for low structural damping). Introducing the probability of a mode to participate in the difference between exact and approximate solution allows us to give an expected value of the factor and then to give statistical bounds on energy considerably reduced when compared to that of the general case. Finally, the method was applied to approximate solutions obtained with a truncated modal expansion; this gave a criterion of validity of the truncated expansion.

Our attention was focused on kinetic energy as this quantity is generally used in vibro-acoustic problems, however, we show that the results can be extended to displacement energy.

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APPENDIX

Let us start with expression (29) of the norm:

$$\|U_i, \tau_{ij}\|^2 = \sum_{n=1}^{\infty}\left(\left|a_n-(1+j\eta)b_n\right|^2/(1+\eta^2)+\left|a_n\frac{\omega_n}{\Omega}-b_n\frac{\Omega}{\omega_n}\right|^2\right)E_n. \tag{A1}$$

For a given displacement field U_i, one minimizes the norm with respect to the stress field σ_{ij}, writing:

$$\frac{\partial}{\partial a_n}\|U_i, \tau_{ij}\|^2 = 0 \quad \forall n \tag{A2}$$

$$\frac{\partial}{\partial a_n^2}\|U_i, \tau_{ij}\|^2 > 0 \quad \forall n. \tag{A3}$$

That is to say:

$$(a_n-(1+j\eta)b_n/1+\eta^2)+\left(a_n\frac{\omega_n}{\Omega}-b_n\frac{\Omega}{\omega_n}\right)\frac{\omega_n}{\Omega} = 0 \quad \forall n \tag{A4}$$

$$\frac{1}{1+\eta^2}+\frac{\omega_n^2}{\Omega} > 0. \tag{A5}$$

Relation (A5) is verified, and then the minimization conditions reduce to

$$a_n = b_n\alpha_n \tag{A6}$$

with

$$\alpha_n = 2+j\eta+\eta^2/(1+\omega_n^2/\Omega^2(1+\eta^2)). \tag{A7}$$

Introducing (A6) in (A1) one obtains:

$$\|U_i, \tau_{ij}\|^2 \geqslant \sum_{n=1}^{\infty}\Gamma_n\frac{\Omega^2}{\omega_n^2}|b_n|^2E_n \tag{A8}$$

with

$$\Gamma_n = \left(\left|\alpha_n-1-j\eta\right|^2/(1+\eta^2)+\left|\alpha_n\frac{\omega_n}{\Omega}-\frac{\Omega}{\omega_n}\right|^2\right)\omega_n^2/\Omega^2. \tag{A9}$$

Inequality (A8) is used in Section 4.

A second lower bound of the norm is obtained minimizing with respect to U_i for a given stress field τ_{ij}.

Taking into account the modal expansion of U_i one writes:

$$\frac{\partial}{\partial b_n}\|U_i, \tau_{ij}\|^2 = 0 \quad \forall n \tag{A10}$$

$$\frac{\partial}{\partial b_n^2}\|U_i, \tau_{ij}\|^2 > 0 \quad \forall n. \tag{A11}$$

After calculation one obtains:

$$\|U_i, \tau_{ij}\|^2 \geqslant \sum_{n=1}^{\infty}\Delta_n|a_n|^2E_n \tag{A12}$$

with:

$$\Delta_n = |1-(1+j\eta)\beta_n|^2/(1+\eta^2)+\left|\frac{\omega_n}{\Omega}-\beta_n\frac{\Omega}{\omega_n}\right|^2 \tag{A13}$$

and

$$\beta_n = (2+j\eta+\eta^2)/(1+2j\eta+\eta^2+(1+\eta^2)\Omega^2/\omega_n^2). \tag{A14}$$

Inequality (A12) is used in Section 5.